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Complete Closest-Target Based Directional FDH Measures of Efficiency in DEA

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Abstract

In this paper, we aim to overcome three major shortcomings of the FDH (Free Disposal Hull) directional distance function through developing two new, named Linear and Fractional CDFDH, complete FDH measures of efficiency. To accomplish this, we integrate the concepts of similarity and FDH directional distance function. We prove that the proposed measures are translation invariant and unit invariant. In addition, we present effective enumeration algorithms to compute them. Our proposed measures have several practical advantages such as: (a) providing closest Pareto-efficient observed targets (b) incorporating the decision maker's preference information into efficiency analysis and (c) being flexible in computer programming. We illustrate the newly developed approach with a real world data set.

Keywords: DEA, FDH, Efficiency, Closest Target, FDH Directional Distance Function.

1. Introduction

Data envelopment analysis (DEA), originally developed by Charnes et al. (1978) and later extended by Banker et al. (1984), is a non-parametric linear programming-based method to evaluate the relative efficiency of a set of homogeneous decision making units (DMUs). The relative comparison in DEA is made with reference to a production possibility set (PPS) constructed from the set of observed DMUs by assuming several postulates. One of the assumed postulates in constructing the conventional DEA PPSs is the convexity assumption. Therefore, in evaluating a given inefficient DMU by the conventional DEA models, the obtained projection point may be a virtual activity (i.e., be an unobserved DMU). However, in most practical applications, virtual activities may have not actual existence. In such cases, the production technology does not satisfy the convexity assumption and, therefore, assuming it is meaningless. By relaxing the convexity assumption, Deprins et al. (1984) proposed an extension of the conventional technologies, called Free Disposal Hull (FDH) non-convex technology. One approach for computing the value of FDH measures is using the common mixed integer linear programming (MILP) techniques. However, solving an MILP is not efficient, from computational point of view. Thus, Tulkens (1993) presented effective enumeration algorithms, based on vector dominance reasoning, to compute the traditional input and output efficiency measures developed by Farrell (1957). Up to now, many papers in DEA literature have been published studying the FDH approach from both application and theoretical perspectives. Some articles relating to the application of the FDH models are Ruiz-Torres and Lopez (2004), Ching-Kuo (2007), Amin and Hosseini Shirvani (2009), Witte et al. (2010), Halkos and Tzeremes (2010), Blancard et al. (2011) and Alimardani et al. (2012). The theoretical aspect of the FDH approach has been also explored in Kerstens and Vanden Eeckaut (1999), Thrall (1999), Cherchye et al.(2000), Agrell and Tind (2001), Cherchye et al. (2001), Leleu (2006, 2009), Keshvari and Dehghan Hardoroudi (2008) and Alirezaee and Khanjani Shiraz (2010). Among the previous researchers, Cherchye et al. (2001) using the directional distance function, recently introduced by Chambers et al. (1996; 1998), developed the FDH directional distance function (we call it DFDH) to estimate the FDH technical inefficiency. They, further, by extending the Tulkens's (1993) algorithms presented an enumeration algorithm for computing the FDH inefficiency of DMUs. As we point out in this paper, the DFDH suffers from a number of major shortcomings including

(i) Not providing an efficiency score for the DMU under evaluation.

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$\vec{D}_T(x, y; -g^-, g^+)$ provides a complete characterization of the technology, i.e.

$$\vec{D}_T(x, y; -g^-, g^+) \geq 0 \iff (x, y) \in T \tag{4}$$

2.3 The FDH Directional Distance Function

By formulating the directional distance function relative to (2), we have the mathematical formulation of the DFDH (Cherchye et al., 2001) as

$$\begin{aligned} \beta^* = & Max \quad \beta \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io} - \beta g_i^-, \quad i = 1, \dots, m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro} + \beta g_r^+, \quad r = 1, \dots, s, \\ & \textstyle\sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \in \{0,1\}, \qquad j = 1, \dots, n, \\ & s_i^- \geq 0, s_r^+ \geq 0, \qquad i = 1, \dots, m, r = 1, \dots, s. \end{aligned} \tag{5}$$

The above model is a mixed integer linear programming (MILP) problem in which the variables λ_j, $j = 1, \dots, n$, are binary variables. In this model, since $\sum_{j=1}^{n} \lambda_j = 1$ and $\lambda_j \in \{0,1\}$ for $j = 1, \dots, n$, only one of the binary variables λ_j ($j = 1, \dots, n$) will be unit valued, while all the other variables will be 0 in the optimal solution. Therefore, in the model (5) the reference point is chosen among the existing DMUs instead of their combinations. In other words, the FDH model (5) compares the under assessment DMU with an observed unit.

The model (5) can be solved via the common MILP techniques. However, Cherchye et al. (2001) proposed an enumeration algorithm to compute the optimal solution for (5). Their algorithm is based on an alternative equivalent characterization of the FDH technology that is provided using monotone hulls of the observed DMUs. The monotone hull of DMUj is defined as follows:

$$M(j) = \{(x, y) | x \geq x_j, y \leq y_j\} \tag{6}$$

From this definition, the FDH technology can be approximated as:

$$T_{FDH} = \cup_{j=1}^{n} M(j) \tag{7}$$

Based on this approximation, the model (5) is rewritten as

$$\beta^* = \text{Max}_{j=1,\dots,n} \left\{ \text{Max} \left\{ \beta \left| \begin{matrix} x_{ij} \leq x_{io} - \beta g_i^-, i = 1, \dots, m \\ y_{rj} \geq y_{ro} + \beta g_r^+, r = 1, \dots, s \end{matrix} \right. \right\} \right\} \tag{8}$$

Since we have suppose that $g^- > 0$ and $g^+ > 0$, so according to Cherchye et al.'s (2001) enumeration algorithm the optimal objective of (5) can be computed as

$$\beta^* = \underset{j=1,\dots,n}{\text{Max}} \{\beta_j\}$$
$$\beta_j = \text{Min}_{\substack{i=1,\dots,m, \\ r=1,\dots,s}} \left\{ \frac{1}{g_i^-} (x_{io} - x_{ij}),\ \frac{1}{g_r^+} (y_{rj} - y_{ro}) \right\}, \forall j \tag{9}$$

3. Complete Directional FDH Measures of Efficiency

As mentioned in the introductory section, the DFDH, (5), suffers from some serious shortcomings. In this section, we thoroughly discuss about these shortcomings and attempt to remedy them, by proposing new solution approaches.

Shortcoming 1 of the DFDH: In the model (5), β^*, in general, cannot be interpreted as an efficiency index for any arbitrary direction vector. A way of avoiding this shortcoming is imposing one of the following primary conditions on the direction vector g

$$\text{Max}_{j=1,\dots,n} \left\{ \frac{x_{io}}{g_i^-} \right\} \leq 1 \tag{10}$$

$$\underset{j=1,\dots,n}{\text{Max}} \left\{ \frac{\overline{x}_i - \underline{x}_i}{g_i^-} \right\} \leq 1, \overline{x}_i = \underset{j=1,\dots,n}{\text{Max}} \{x_{ij}\}, \underline{x}_i = \text{Min}_{j=1,\dots,n} \{x_{ij}\} \tag{11}$$

which assures that $\beta^* \leq 1$ and, thereby, $1 - \beta^*$ can be interpreted as an efficiency index. For example, each of the following direction vectors satisfies the condition (10):

$$g_i^- = x_{io}, g_r^+ = y_{ro}, i = 1, \dots, m, r = 1, \dots, s \tag{12}$$

$$\begin{cases} g_i^- = \overline{x}_i = \text{Max}_{j=1,\dots,n} \{x_{ij}\}, \forall\, i, \\ g_r^+ = \overline{y}_r = \text{Max}_{j=1,\dots,n} \{y_{rj}\}, \forall\, r \end{cases} \tag{13}$$

In addition, the following direction vector satisfies the condition (11):

$$\begin{cases} g_i^- = \overline{x}_i - \underline{x}_i = \text{Max}_{j=1,\dots,n} \{x_{ij}\} - \text{Min}_{j=1,\dots,n} \{x_{ij}\}, \forall\, i, \\ g_r^+ = \overline{y}_r - \underline{y}_r = \text{Max}_{j=1,\dots,n} \{y_{rj}\} - \text{Min}_{j=1,\dots,n} \{y_{rj}\}, \forall\, r \end{cases} \tag{14}$$

3.1 Complete Furthest-Target Based Directional FDH Measures of Efficiency

Shortcomings 2 of the DFDH: The model (5) radially (proportionately) projects the given DMU onto the frontier of T_{FDH}. Thus, the DFDH fails to take account the non-zero input and output slacks as sources of inefficiency that makes the projected point not to be necessarily Pareto-efficient. To create a visual representation, consider the FDH technology depicted in Fig 1. This technology is constructed by the Pareto-efficient, A, B, C, D, and inefficient, E and F, DMUs for the simplest case of single input and single output. As evident in the figure, maximum proportional decrease in input and increase in output of the inefficient unit F in the direction of g is achieved on M(C). By these improvements, the unit F is projected onto the boundary unobserved point F', and the efficient unit C is determined as the reference DMU to the unit F. However, the value of s^- is not contributed in evaluation of the unit F.

Relative to the radial DEA models, the non-radial ones have higher discriminatory power in evaluating DMUs. In addition, as noted by Silva et al. (2003), "the non-convex

Linear CDFDH	Fractional CDFDH
$\tau^C = Min \ \frac{1}{m}\sum_{i=1}^{m}\beta_i^- + \frac{1}{s}\sum_{r=1}^{s}\beta_r^+$	$\rho_F^C = Max \ \frac{1-\frac{1}{m}\sum_{i=1}^{m}\beta_i^-}{1+\frac{1}{s}\sum_{r=1}^{s}\beta_r^+}$
$s.t. \ \sum_{j\in E}\lambda_j x_{ij} = x_{io} - \beta_i^- g_i^-, \quad i = 1,\dots,m,$	$s.t. \ \sum_{j\in E}\lambda_j x_{ij} = x_{io} - \beta_i^- g_i^-, \quad i = 1,\dots,m,$
$\sum_{j\in E}\lambda_j y_{rj} = y_{ro} + \beta_r^+ g_r^+, \quad r = 1,\dots,s,$	$\sum_{j\in E}\lambda_j y_{rj} = y_{ro} + \beta_r^+ g_r^+, \quad r = 1,\dots,s,$
$\sum_{j\in E}\lambda_j = 1,$	$\sum_{j\in E}\lambda_j = 1,$
$\lambda_j \in \{0,1\}, \quad j = 1,\dots,n,$	$\lambda_j \in \{0,1\}, \quad j = 1,\dots,n,$
$\beta_i^- \geq 0, \beta_r^+ \geq 0, \quad i = 1,\dots,m, r = 1,\dots,s.$	$\beta_i^- \geq 0, \beta_r^+ \geq 0, \quad i = 1,\dots,m, r = 1,\dots,s.$
(18)	(19)

In the above models, E is the set of all F-efficient (L-efficient) DMUs. The guiding idea of formulating the models (18) and (19), to determine the closest targets, is minimizing the distance of the given inefficient DMUo from all the F-efficient (L-efficient) DMUs dominating DMUo. Corresponding to the models (18) and (19), we introduce two new efficiency indices, ρ_F^C and ρ_L^C, where the former is defined as follows:

$$\rho_L^C := \left[1-\frac{1}{m}\sum_{i=1}^{m}\beta_i^{-*}\right] \times \left[1+\frac{1}{s}\sum_{r=1}^{s}\beta_r^{+*}\right]^{-1} \quad (20)$$

Similar to ρ_F and ρ_L, the indexes ρ_F^C and ρ_L^C satisfy the property of efficiency requirement and can be interpreted as efficiency measures. Relative to these measures, we define the "efficiency" as follows:

Definition 1. DMU_o is said to be FC-efficient (LC-efficient) if and only if $\rho_F^C = 1$ ($\rho_L^C = 1$).

The following relationships are held among the proposed models:

(R1). DMU_o is F-efficient if and only if it is L-efficient, i.e., $\rho_F = 1$ if and only if $\rho_L = 1$.

(R2). DMU_o is FC-efficient if and only if it is LC-efficient, i.e., $\rho_F^C = 1$ if and only if $\rho_L^C = 1$.

(R3). DMU_o is Pareto-efficient if and only if $\rho_F = 1$ if and only if $\rho_F^C = 1$.

(R4). $\rho_F \leq \rho_L$, $\rho_L^C \leq \rho_F^C$ and $\rho_F \leq \rho_F^C$.

3.3 Properties of the Linear and Fractional DFDH / CDFDH Models

In this section, we study the properties of the proposed models.

P1. Completeness (Cooper et al., 1999)

Our proposed measures are "complete", in the sense that they are non-oriented and take account all the non-zero slacks as sources of inefficiency.

P2. Straightforward interpretation

Each of the efficiency indexes ρ_L, ρ_F, ρ_L^C and ρ_F^C can be interpreted as the product of two separate components of the input efficiency, $\theta_I = 1 - \frac{1}{m}\sum_{i=1}^{m}\beta_i^{-*}$, and the output efficiency, $\theta_O = \left[1+\frac{1}{s}\sum_{r=1}^{s}\beta_r^{+*}\right]^{-1}$. This interpretation gives a better explanation of the efficiency of the under assessment DMU.

P3. Translation invariance (Ali and Seiford, 1990; Lovell and Pastor, 1995; Pastor, 1996; Cooper et al., 1999)

The property of translation invariance assures that translating the original input and output data has no influence on the optimal solutions. By choosing the direction vector (14), the models (15), (16), (18) and (19) will be translation invariant.

To demonstrate this, suppose that $\tilde{x}_{ij} = x_{ij} + v_i$ $(i = 1,\dots,m)$ and $\tilde{y}_{rj} = y_{rj} + u_r$ $(r = 1,\dots,s)$, respectively indicate the translated ith input and rth output of DMUj. Obviously, the original direction (14) will not be changed for the translated data. Therefore, from the constraint $\sum_j \lambda_j = 1$, it follows that our models are translation invariant.

This property makes our models free from the primary positivity assumption on the data whereby they are able to appropriately deal with negative data. In this case, by adding suitable constants to the affected input or output rows we can transform them to positive valued data and use the transformed data in our models.

P4. Units invariance (Cooper et al., 1999; Lovell and Pastor, 1995)

By choosing a direction vector such that g_i^- and g_r^+ respectively have the same units of measurement as the ith input and the rth output, the models (15), (16), (18) and (19) will be unit invariant. Since, g_i^- and the ith input have the same units of measurement, when we rescale the ith input by the scalar $\alpha > 0$, the ith component of the direction vector is converted to αg_i^-, accordingly. Observe that, by multiplying the constraint $\sum_j \lambda_j(\alpha x_{ij}) = \alpha x_{io} - \beta_i^-(\alpha g_i^-)$ by $\frac{1}{\alpha}$, the same constraint of the original model is obtained. For example, the vectors (12) and (13) satisfy this property.

This property indicates that we can rescale each input or output with an arbitrary scalar, without any affection on the optimal solutions of our models.

P5. Alternative optima invariance (Cooper et al., 1999)

5. Real World Example

Example1. To provide an application of the proposed approach, we discuss the efficiency assessment of 15 university departments, denoted by D1, D2,…, D15, with three inputs and three outputs as follows:

Table 2 shows the input-output data, taken from Soleimani-damaneh and Mostafaee (2009).
Table 3 represents the efficiency scores, projection points, and reference points obtained from the DFDH, Fractional FDFDH, and Fractional CDFDH models where the direction vector (13) is used in them.

Table 1
Titles of inputs and outputs of Example 2

Input1:	Budget	**Output1**	Average of the scores of the students,
Input2:	Area of cultural, educational, and research space	**Output2**	Number of the students accepted for graduate studies (in the past two years),
Input3:	Number of books in the library	**Output3**	Satisfaction of students, staff, and professors

Table 2
Data related to the application with real world data (Example 1)

	I_1	I_2	I_3	O_1	O_2	O_3
D1	26	20	10	15.5	8	26
D2	20	15	7	17.2	5	25
D3	22	10	9.5	14.3	8	23
D4	15	12	8.4	14	5	20
D5	30	22	10	12	3	20
D6	35	15	11	16.3	4	22
D7	35	25	12	12	2	18
D8	34	24	12	12	2	19
D9	20	16	9.5	14.3	10	28
D10	22	17	10	13.5	8	26
D11	24	19	8	15	9	29
D12	18	20	12	16	8	23
D13	28	20	10.1	14.2	6	20
D14	30	12	9	14	5	20
D15	25	15	10	17	3	29

Table 3
Data related to the application with real world data (Example 2)

	DFDH			Fractional FDFDH			Fractional CDFDH		
	Score	*Proj.*	*Ref.*	*Score*	*Proj.*	*Ref.*	*Score*	*Proj.*	*Ref.*
D1	1.0000	D1	D1	1.0000	D1	D1	1.0000	D1	D1
D2	1.0000	D2	D2	1.0000	D2	D2	1.0000	D2	D2
D3	1.0000	D3	D3	1.0000	D3	D3	1.0000	D3	D3
D4	1.0000	D4	D4	1.0000	D4	D4	1.0000	D4	D4
D5	0.8276	P5	D2	0.5919	D9	D9	0.7175	D1	D1
D6	1.0000	D6	D2	0.6874	D2	D2	0.6874	D2	D2
D7	0.7586	P7	D2	0.4565	D2	D2	0.5825	D1	D1
D8	0.7931	P8	D2	0.4787	D2	D2	0.6044	D1	D1
D9	1.0000	D9	D9	1.0000	D9	D9	1.0000	D9	D9
D10	0.9600	P10	D9	0.8630	D9	D9	0.8630	D9	D9
D11	1.0000	D11	D11	1.0000	D11	D11	1.0000	D11	D11
D12	1.0000	D12	D12	1.0000	D12	D12	1.0000	D12	D12
D13	0.9600	P13	D11	0.6957	D9	D9	0.8427	D1	D1
D14	1.0000	D14	D4	0.8405	D4	D4	0.8405	D4	D4
D15	1.0000	D15	D15	1.0000	D15	D15	1.0000	D15	D15

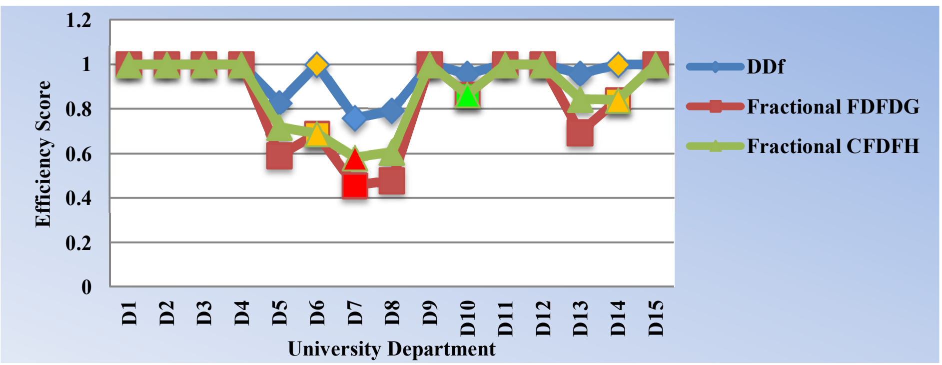

Fig. 2. Comparison of departments' efficiencies

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